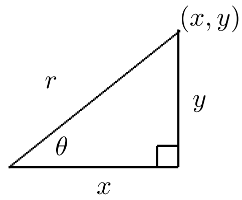
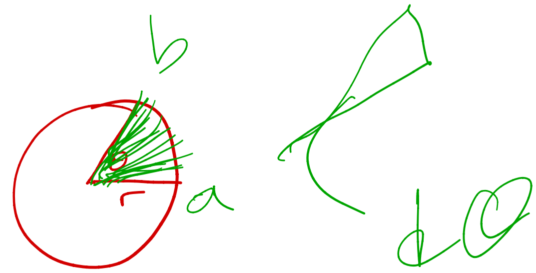
Calc BC Polar WS1 Name: Block: Seat:



Polar Functions and Equations



(x, y) vs (r, θ)
$x^2 + y^2 = r^2$
$x = r\cos\theta$
$y = r\sin\theta$
$\theta = \arctan\frac{y}{x}$
Area$= \frac{1}{2}r^2\theta$

$\left(\frac{\theta}{2\pi}\right)\pi r^2$

$A_{s.} = \frac{1}{2}\theta r^2$

$\frac{1}{2}\int_\alpha^\beta r^2\, d\theta$

The following formulas will be used on the AP Calculus exam. You need to know them.

The first derivative (the change in y with respect to x)	$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{(r\sin\theta)'}{(r\cos\theta)'}$
so if $r = f(\theta)$, using the product rule, the complete polar form is	$= \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$ or $\frac{r\cos\theta + r'\sin\theta}{r'\cos\theta - r\sin\theta}$
The Area **INSIDE** a polar curve is given by	$\frac{1}{2}\int_\alpha^\beta r^2\ d\theta$
To find the area **BETWEEN** polar curves, sketch a wedge-shaped $d\theta$ and find the angle coordinates of the points of intersection. The area will be	$\frac{1}{2}\int_\alpha^\beta R^2 - r^2\ d\theta$ or $\frac{1}{2}\int_\alpha^\beta (OR)^2 - (IR)^2\ d\theta$
The length along the arc of a polar curve is given by	$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$

1. Find the tangent line (in Cartesian $y - y_1 = m(x - x_1)$ form) to the curve $r = \cos(2\theta)$ at $\theta = \frac{\pi}{4}$.

   (a) Find equations for x and y with respect to θ by substituting $r = \cos(2\theta)$ into the generic $x = r\cos\theta$ and $y = r\sin\theta$ equations.

   $$r = \cos(2\theta)$$
   $$x = (\cos 2\theta)\cos\theta \Big|_{\theta=\frac{\pi}{4}}$$
   $$y = (\cos 2\theta)\sin\theta \Big|_{\theta=\frac{\pi}{4}}$$

   (b) Find $x'(\theta)$ and $y'(\theta)$

   $$\begin{array}{c|c} \cos 2\theta & \cos\theta \\ \hline -2\sin 2\theta & -\sin\theta \end{array} \qquad \begin{array}{c|c} \cos 2\theta & \sin\theta \\ \hline -2\sin 2\theta & \cos\theta \end{array}$$

   $$x'(\theta) = -\sin\theta\cos 2\theta - 2\sin 2\theta\cos\theta$$
   $$y'(\theta) = \cos\theta\cos 2\theta - 2\sin 2\theta\sin\theta$$

   (c) Find the slope

   $$\frac{dy}{dx}\Big|_{\theta=\frac{\pi}{4}} = \frac{\left(\frac{\sqrt{2}}{\cancel{2}}\right)(0) - (\cancel{2})\left(\frac{\sqrt{2}}{\cancel{2}}\right)}{\left(-\frac{\sqrt{2}}{\cancel{2}}\right)(0) - (\cancel{2})\left(\frac{\sqrt{2}}{\cancel{2}}\right)} = \frac{-\sqrt{2}}{-\sqrt{2}} = 1$$

   (d) Find the (x, y) coordinate when $\theta = \frac{\pi}{4}$

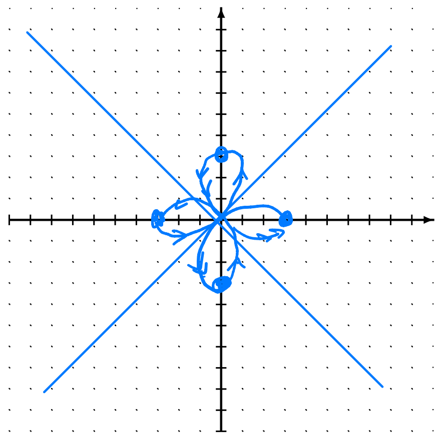
   $(0,0)$

   $y = x$

   (e) What is the tangent line?

   $y = x$

2. What are the equations of all the tangent lines at the pole of $r = 3\cos(2\theta)$?



$\cancel{3}\cos 2\theta = \frac{0}{3}$

$2\theta = \cos^{-1}(0)$

$2\theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$\theta = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

$$\frac{dy}{dx} = \frac{r\cos\theta + r'\sin\theta}{r'\cos\theta - r\sin\theta}$$

$r = 3\cos(2\theta)$ $\qquad r' = -6\sin(2\theta)$

$\theta = \frac{\pi}{4}$

$$\left.\frac{\cancel{r}\frac{\sqrt{2}}{2} + r'\frac{\sqrt{2}}{2}}{r'\frac{\sqrt{2}}{2} - \cancel{r}\frac{\sqrt{2}}{2}}\right|_{r} = \frac{-3\sqrt{2}}{-3\sqrt{2}} = 1$$

Area Sect $\left(\frac{\theta}{2\pi}\right)\pi r^2$

3. Consider $r = 2 + 2\cos(\theta)$

$\frac{1}{2}\theta r^2$

$\sin^2\theta + \cos^2\theta = 1$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta + \sin^2\theta = \cos^2\theta$

$\cos 2\theta + (1 - \cos^2\theta) = \cos^2\theta$

$\frac{\cos 2\theta + 1}{2} = \frac{\cancel{2}\cos^2\theta}{\cancel{2}}$

$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

$\frac{1}{2}\int_\alpha^\beta r^2 \, d\theta$

$\frac{1}{2}\int_0^{2\pi} (2 + 2\cos\theta)^2 \, d\theta$

$\frac{1}{2}\int_0^{2\pi} 4 + 8\cos\theta + 4\cos^2\theta \, d\theta$

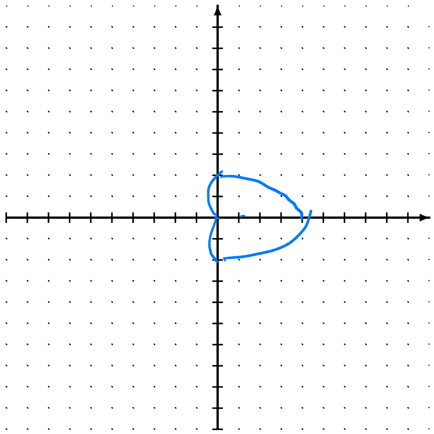
$\frac{1}{2}\left[4\theta\right]_0^{2\pi} + \frac{1}{2}\left[8\sin\theta\right]_0^{2\pi} + \frac{1}{\cancel{2}}\int \cancel{4}^{2}\frac{(1+\cos 2\theta)}{\cancel{2}} \, d\theta$

$4\pi + 0 + \left[\theta - \frac{\sin 2\theta}{2}\right|_0^{2\pi}$

$4\pi + 0 + 2\pi$

6π

(a) Graph



(b) Find the area inside

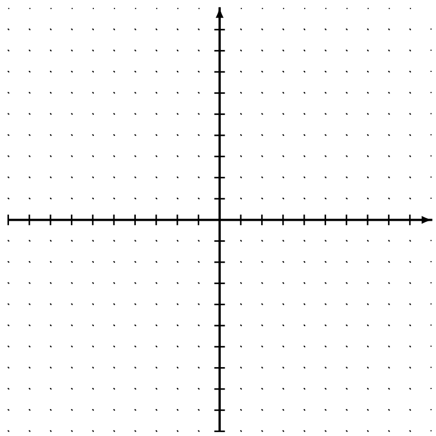
(c) Find $\frac{dr}{d\theta}$ $= 0 - 2\sin\theta$

(d) Find the length of the curve (Beware, in this case one revolution is $0 \leq \theta \leq 2\pi$ or $\pi \leq \theta \leq 3\pi$, but that is not always the case–set $r = 0$ and solve for θ)

$\int_0^{2\pi} \sqrt{(2 + 2\cos\theta)^2 + 4\sin^2\theta} \, d\theta$

$= 16$

4. Consider the rose $r(\theta) = 2\cos(2\theta)$ (Would a rose by any other equation smell as sweet?)

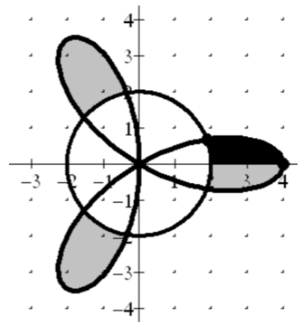


   (a) Graph

   (b) For what values of θ is $r = 0$?

   (c) Find the area of one of the four petals (Write the integral, then write the calculator result).

   (d) What is the area inside the complete rose?



5. Find the area inside of $r_1 = 4\cos(3\theta)$ and outside $r_2 = 2$. You could calculate the darkest area and multiply by 6, or else calculate the area of one shaded region and multiply by 3. In either case the tricky bit is find the limits of integration where the curve intersect.

   (a) Find where $4\cos(3\theta) = 2$ graphically and analytically.

   (b) Set up the integral and use the calculator to find the area.

6. Given $r = 3 + 3\sin\theta$, find the distance of the arc when the curve is in quadrants I and II

7. Find the are enclosed by one petal of $r = 5\sin(2\theta)$

8. Find the area of the region inside the circle $r = 4$ and outside $r = 2 + 2\sin\theta$